Due Sun

4.1 – Real Vector Spaces

as opposed to complex

Definition: (Vector space)

Let V be an arbitrary nonempty set of objects for which two operations are defined: addition and multiplication by numbers called **scalars**. By **addition** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the **sum** of $\mathbf{u}$ and $\mathbf{v}$; by **scalar multiplication** we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the **scalar multiple** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k and m, then we call V a **vector space** and we call the objects in V **vectors**.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V. Closure under addition
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There exists an object in V, called the **zero vector**, that is denoted by $\mathbf{0}$ and has the property that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a **negative** of $\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is an object in V, then $k\mathbf{u}$ is in V. closed under scalar mult.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

Notes: ① vector addition and scalar mult. can be any rules that satisfy these axioms

② A vector is any object in a vector space.

#6 Determine whether the set equipped with the given operations is a vector space. For those that are not vector spaces, Identify the vector space axioms that fail.

The set of all n-tuples of real numbers that have the form $(x, x, \ldots, x)$ with the standard operations on R^n.

Let $\vec{u} = (u, u, \ldots, u)$, $\vec{v} = (v, v, \ldots, v) \in R^n$

* ① $\vec{u} + \vec{v} = (u+v, u+v, \ldots, u+v)$ ✓

②, ③, ⑦ - ⑩ are satisfied by prop. of real #s and Thm 3.1.1.

* ⑥ $k\vec{u} = (ku, ku, \ldots, ku)$ ✓

④ $\vec{0} = (0, 0, \ldots, 0)$ ✓ ⑤ $-\vec{u} = (-u, -u, \ldots, -u)$

this is a vector space.

Examples of Vector Spaces

1. The simplest vector space is $\{\mathbf{0}\}$.
2. R^n with the usual operations of addition and scalar multiplication of n-tuples
3. R^∞, the set of infinite sequences of numbers with the usual operations of addition and scalar multiplication performed componentwise
4. The set of $m \times n$ matrices with matrix addition and scalar multiplication, denoted M_{mn}
5. The set of real-valued functions $f(x)$ that are defined for all $x \in R$, denoted $F(-\infty, \infty)$ with addition and scalar multiplication $(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x)$ and $(k\mathbf{f})(x) = kf(x)$
6. The set of polynomials of degree $\leq n$, denoted P_n

Example: Show why #5 is a vector space.

$k \in \mathbb{R}$

Let $\vec{f}$ and $\vec{g} \in F(-\infty,\infty)$ and k a scalar

① $(\vec{f}+\vec{g})(x) = f(x)+g(x) = (f+g)(x)$ ✓

⑥ $(k\vec{f})(x) = kf(x) = k\vec{f}(x)$

At any point x, $\vec{f}(x)$ & $\vec{g}(x)$ are real #s;

thus 2, 3, 7, 8, 9, 10 are satisfied

by properties of real #s.

4. $\vec{0}$ is the function $f(x)=0$.

5. $-\vec{f}(x) = -f(x) \in F(-\infty,\infty)$ for

all $\vec{f} \in F(-\infty,\infty)$.

set: $\mathbb{R}^2$

std addition

Example 4.1.7: Not a vector space

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. Define $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$ and $k\mathbf{u} = (ku_1, 0)$.

⑩ Let $k=1$. Then $1\vec{u} = (u_1, 0) \neq \vec{u}$

Not a vector space.

Example 4.1.8: Unusual vector space

Let V be the set of positive real numbers, so $\mathbf{u} = u$ and $\mathbf{v} = v$ are positive real numbers. Define $\mathbf{u} + \mathbf{v} = uv$ and $k\mathbf{u} = u^k$.

vector addition is mult.

scalar mult is exponentiation

For instance, $\vec{2} + \vec{5} = 10$, $2(\vec{5}) = 5^2 = 25$

①, ⑥ ✓ both ops. result in pos #s

②, ③ ✓ mult. of real #s is comm. and assoc.

④ The zero element: $\vec{0} = 1$ because

$\vec{0} + \vec{u} = 1u = u = \vec{u}$

⑤ The negative of a vector $\vec{u}$ is its reciprocal: $-1\vec{u} = u^{-1} = \frac{1}{u}$, and

$\vec{u} + \vec{\frac{1}{u}} = u\left(\frac{1}{u}\right) = 1 = \vec{0}$

⑦ $k(\vec{u} + \vec{v}) = (uv)^k = u^k v^k = k\vec{u} + k\vec{v}$ ✓

⑧ $(k+m)\vec{u} = u^{k+m} = u^k u^m = k\vec{u} + m\vec{u}$ ✓

⑨ $k(m\vec{u}) = (u^m)^k = u^{mk} = u^{km} = (km)\vec{u}$

⑩ $1\vec{u} = u^1 = u = \vec{u}$ ✓

Example: Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$\mathbf{u} + \mathbf{v} = (u_1 + v_1 + 2,\ u_2 + v_2)$, $k\mathbf{u} = (ku_1, ku_2)$.

a. Show that Axioms 4 and 5 hold.

b. Find all axioms that fail to hold.

4. Let $\vec{0} = (a, b)$: $\vec{0} + \vec{u} = (a, b) + (u_1, u_2)$

$= (a + u_1 + 2,\ b + u_2) = (u_1, u_2) \Rightarrow a = -2,\ b = 0$

$\vec{0} = (-2, 0)$

5. Let $-\vec{u} = (a, b)$. $\vec{u} + (-\vec{u}) = \vec{0}$

$(u_1 + a + 2,\ u_2 + b) = (-2, 0)$

$a = -u_1 - 4,\ b = -u_2 \Rightarrow -\vec{u} = (-u_1 - 4, -u_2)$

7 fails : $k(\vec{u} + \vec{v}) = k(u_1 + v_1 + 2,\ u_2 + v_2)$

$= (ku_1 + kv_1 + 2k,\ ku_2 + kv_2)$

$k\vec{u} = (ku_1, ku_2)$, $k\vec{v} = (kv_1, kv_2)$, so

$k\vec{u} + k\vec{v} = (ku_1 + kv_1 + 2,\ ku_2 + kv_2)$

8 similarly fails.

Theorem 4.1.1 Let V be a vector space, $\mathbf{u}$ be a vector in V, and k a scalar; then:

a) $0\mathbf{u} = \mathbf{0}$ ✓

b) $k\mathbf{0} = \mathbf{0}$ ✓

c) $(-1)\mathbf{u} = -\mathbf{u}$

d) If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$

Suppose $k\vec{u} = \vec{0}$.

If $k = 0$, $0\vec{u} = \vec{0}$ Done

If $k \neq 0$, $k\vec{u} = \vec{0} \Rightarrow \vec{u} = \frac{1}{k}\vec{0}$

$= \vec{0}$

Example: Prove that the zero vector in any vector space is unique.

Pf: Let V be a vector space and let $\vec{0}$ be a vector such that $\vec{0}+\vec{u}=\vec{u}+\vec{0}=\vec{u}$ for all $\vec{u}\in V$. Suppose there is $\vec{0}'$ $\ni$

$$\vec{0}'+\vec{u}=\vec{u}+\vec{0}'=\vec{u} \quad \forall\ \vec{u}\in V.$$

Let $\vec{u}=\vec{0}$. $\quad \underline{\vec{0}'+\vec{0}=\vec{0}}$

But $\underline{\vec{0}'+\vec{0}=\vec{0}'}$ $\Rightarrow$ $\vec{0}'=\vec{0}$ ✓

OR

Let $\vec{0}\in V$ be such that $\vec{0}+\vec{u}=\vec{u}$ and $\vec{u}+\vec{0}=\vec{u}$ $\forall\ \vec{u}\in V$ and $\underline{\vec{u}+(-\vec{u})=\vec{0}}$.

And let $\vec{0}'\in V \ni \vec{0}'+\vec{u}=\vec{u}+\vec{0}'=\vec{u}$

Since $\vec{0}'+\vec{u}=\vec{u}$ and $\vec{0}+\vec{u}=\vec{u}$,

$$\vec{0}'+\vec{u}=\vec{0}+\vec{u}$$

$$\vec{0}'+\vec{u}+(-\vec{u})=\vec{0}+\vec{u}+(-\vec{u})$$

$$\vec{0}'+\underline{\vec{0}}=\vec{0}+\vec{0}$$

$$\vec{0}'=\vec{0} \quad ✓$$